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Generalized Coherent States for q-Deformed Hyperbolic Pöshel-Teller Potential

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Abstract In this paper, we solve the Schrödinger equation for q-deformed hyperbolic Pöshel-Teller (PT) potential and we obtain the wave function and ladder operators for it. We show that these operators satisfy commutation relations of su(2) Lie algebra. Then we build the generalized coherent states for this q-deformed potential. We show that for the case $q = 1$, we can obtain the same generalized coherent states for usual hyperbolic PT potential.

Keywords Generalized coherent states · Ladder operators · Pöshel-Teller potential

1 Introduction

The standard coherent state system is intimately related to a group [1–3], considered first by Weyl, the so-called Heisenberg-Weyl group. The coherent state method is particularly effective in cases where the Heisenberg-Weyl group is the dynamical symmetry group of a considered physical system. The simplest example is a quantum oscillator under the action of a variable external driving force [2]. In this case the Heisenberg equations of motion coincide with the corresponding equations for the classical variables. In the course of the time evolution, any coherent state remains coherent [2, 4, 5], and the motion of the phase space point representing the coherent state is described by the classical equation. This fact enables one to simplify the quantum problem significantly, reducing it to the corresponding classical problem. The Heisenberg-Weyl group, of course, is not the universal dynamical symmetry group, other symmetry groups appear in many cases. For instance, the symmetry group for spin precession in a variable magnetic field is SU(2) group, and for the problem of a quantum oscillator with variable frequency the symmetry group is SU(1, 1).

In [1, 3, 6, 7], general coherent systems related to representations of an arbitrary Lie group were constructed and investigated, elaborate methods of group theory were employed

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to study properties of these systems. Generalized coherent state, which were introduced in [1, 2], are relevant to an arbitrary Lie group [8, 9], they are parametric by point of homogenous spaces where the group acts. In same cases, one can consider these spaces as generalized phase spaces for classical dynamical systems. The generalized coherent states were also found to be useful in a number of purely mathematical problems, in particular, in the theory of representation of Lie group, as well as in the investigation of special function.

In the recent years, Lie algebra method have been the subject of the interest in many field of physics [10–13].

Schrödinger equation with hyperbolic (sometime called modified) Pöshel-Teller (PT) potential has been calculated with factorization method [14–16], and shown that the achieved operators, satisfied su(2) Lie algebra. We q-deforme this potential using the approach by Arai [17–20], we obtain the normalized wave function for q-deformed hyperbolic Pöshel-Teller (PT) potential and establish the creation and annihilation operators directly from the eigenfunctions for this system, using the factorization method. After that, we show that these operators construct the dynamical algebra su(2). Finally we obtain the generalized coherent states for this potential. We show that, for the case $q=1$, we can obtain the same generalized coherent states for usual hyperbolic PT potential [13, 14].

2 Generalized Coherent States Based on su(2) Algebra

We note that Lie algebra corresponding to the Lie group SU(2) has three generators, $\hat{J}_1$, $\hat{J}_2$ and $\hat{J}_3$, or $\hat{J}_+$, $\hat{J}_-$ and $\hat{J}_0$ as its basis elements, the commutation relations of su(2) Lie algebra is given by [1, 2, 9]:

$$[\hat{J}_0, \hat{J}_+] = \hat{J}_+, \qquad [\hat{J}_0, \hat{J}_-] = -\hat{J}_-, \qquad [\hat{J}_+, \hat{J}_-] = 2\hat{J}_0. \tag{1}$$

The Fock space on which $\{\hat{J}_+, \hat{J}_-, \hat{J}_3\}$ acts, is $\mathbf{H_J} \equiv \{|J,n\rangle | 0 \le n \le 2J\}$ and whose actions are

$$J_+|J,n\rangle = \sqrt{(n+1)(2J-n)}|J,n+1\rangle, \tag{2}$$

$$J_-|J,n\rangle = \sqrt{n(2J-n+1)}|J,n-1\rangle, \tag{3}$$

$$J_0|J,n\rangle = (-J+n)|J,n\rangle, \tag{4}$$

where $|J,0\rangle$ is a normalized state:

$$J_-|J,0\rangle = 0, \tag{5}$$

$$\langle J,0 \mid J,0\rangle = 1. \tag{6}$$

From (2), (3) and (4), we have

$$|J,n\rangle = \frac{(J_+)^n}{\sqrt{(n!)_{2J}P_n}}|J,0\rangle, \tag{7}$$

where ${}_{2J}P_n = (2J)(2J-1)(2J-n+1)$. Now we would like to consider the displaced operator associated to the su(2) Lie algebra, this operator is given by following relation [1, 9]:

$$D(\xi) = e^{\xi J_+ - \bar{\xi} J_-} = \exp(\zeta J_+)\exp(\eta J_3)\exp(\zeta' J_-), \tag{8}$$

where

$$\zeta = \frac{\xi \tanh(|\xi|)}{|\xi|}, \qquad \eta = \log(1+|\zeta|^2), \qquad \zeta' = -\overline{\zeta}, \tag{9}$$

here ξ is a complex number. The displaced operator is the key formula for usual generalized coherent operators. Applying the displacement operator $D(\xi)$ on the state vector $|\psi_0\rangle$, we obtain another representation for the coherent states:

$$|\zeta\rangle = \frac{1}{(1+|\zeta|^2)^J} \exp(\zeta J_+)|0\rangle, \tag{10}$$

where $|J,0\rangle = |0\rangle$.

Expanding the exponential function, we obtain the decomposition of the coherent state over the orthogonal basis as [9]

$$|\zeta\rangle = \frac{1}{(1+|\zeta|^2)^J} \sum_{n=0}^{2j} \sqrt{\frac{{}_{2J}P_n}{n!}} \zeta^n |J,n\rangle. \tag{11}$$

The above equation is the generalized coherent state of su(2) Lie algebra, we will use this relation in Sect. 4.

3 Schrödinger Equation with q-Deformed Hyperbolic PT Potential

According to Ref. [13], the hyperbolic PT potential is given by

$$V(x) = -\frac{D}{\cosh^2(\alpha x)}, \qquad -\frac{\pi}{2} < x < \frac{\pi}{2}, \tag{12}$$

where D specifies the well depth of the potential. Here we will assume $D > 0$ and similar to [17, 18], we can consider q-deformed hyperbolic PT potential as [17, 18],

$$V(x) = -\frac{D}{\cosh_q^2(\alpha x)}, \tag{13}$$

we applied the deformed hyperbolic functions introduced for first time by Arai [18],

$$\cosh_q(\alpha x) = \frac{e^{\alpha x} + q e^{-\alpha x}}{2}, \tag{14}$$

$$\sinh_q(\alpha x) = \frac{e^{\alpha x} - q e^{-\alpha x}}{2}, \tag{15}$$

where $q > 0$ is a real parameter. When q is complex, we call the above deformed hyperbolic functions as the generalized deformed (q-deformed) hyperbolic functions [19, 20]. By substituting the above potential in Schrödinger equation, we have

$$\frac{d^2\psi_n(x)}{dx^2} + \frac{2\mu}{\hbar^2}\left(E + \frac{D}{\cosh_q^2(\alpha x)}\right)\psi_n(x) = 0. \tag{16}$$

Then by performing the change of variable as $u = \tanh_q(\alpha x)$ and introducing the following parameters,

$$\epsilon = \sqrt{\frac{-2\mu E}{\alpha^2\hbar^2}}, \qquad \beta(\beta+1) = \frac{2\mu D}{\alpha^2\hbar^2}, \tag{17}$$

we can rewrite (16) as

$$\frac{d}{du}\left[(1-u^2)\frac{d\psi_n(u)}{du}\right] + \left[\frac{\beta(\beta+1)}{q} - \frac{\epsilon^2}{1-u^2}\right]\psi_n(u) = 0. \tag{18}$$

We take the wave function with the following form

$$\psi_n(u) = (1-u^2)^{\epsilon/2} w(u), \tag{19}$$

and with change of variable u as

$$\xi = \frac{1}{2}(1-u), \tag{20}$$

we obtain

$$\xi(1-\xi)\frac{d^2 w(\xi)}{d\xi^2} + (\epsilon+1)(1-2\epsilon)\frac{dw(\xi)}{d\xi} - \epsilon(\epsilon+1)w(\xi) + \frac{\beta(\beta+1)}{q}w(\xi) = 0. \tag{21}$$

By comparing the above equation with the following hypergeometric differential equation

$$x(1-x)\frac{d^2y}{dx^2} + [c-(a+b+1)x]\frac{dy}{dx} - aby = 0, \tag{22}$$

we can introduce

$$c = \epsilon + 1, \qquad a+b+1 = 2(\epsilon+1), \tag{23}$$

and

$$a = \epsilon + \frac{1}{2} - \sqrt{\frac{1}{4} + \frac{\beta(\beta+1)}{q}}, \qquad b = \epsilon + \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{\beta(\beta+1)}{q}}. \tag{24}$$

Now, we can write the wave function as

$$\psi_n(u) = N_n(1-u^2)^{\epsilon/2}\,{}_1F_1\left(a, b, \epsilon+1; \frac{1-u}{2}\right), \tag{25}$$

where ${}_1F_1(a, b, \epsilon+1; \frac{1-u}{2})$ is the hyperopic function. From condition of the finiteness of the above wave function, we obtain the general quantum condition as:

$$a = -n, \quad n = 0, 1, 2, \ldots, \tag{26}$$

then the wave function turns to

$$\psi_n(u) = N_n(1-u^2)^{\epsilon/2}\,{}_1F_1\left(-n, -n+2s, \frac{1}{2}+s-n; \frac{(1-u)}{2}\right), \tag{27}$$

where

$$s=\sqrt{\frac{1}{4}+\frac{\beta(\beta+1)}{q}}. \tag{28}$$

By considering the relation between hypergeometric and Gegenbauer polynomials as [15]

$$C_n^{\lambda}(x)=\frac{\Gamma(2\lambda+n)}{n!\Gamma(2\lambda)}\,{}_1F_1\left(-n,2\lambda+n,\frac{1}{2}+\lambda;\frac{(1-x)}{2}\right), \tag{29}$$

the wave function can be written as

$$\psi_n(u)=N_n(1-u^2)^{\epsilon/2}C_n^{-n+s}(u), \tag{30}$$

where N_n is the normalization factor. By using the following integral relation

$$\int_{-1}^{1}(1-x^2)^{\beta-3/2}(1+x)[C_n^{\beta}(x)]^2dx=\frac{\pi^{1/2}\Gamma(\beta-1/2)\Gamma(2\beta+n)}{n!\Gamma(\beta)\Gamma(2\beta)}, \tag{31}$$

we achieve to the normalization factor as:

$$N_n=\sqrt{\frac{\alpha n!(-n+s+1!)(-2n+2s-1)!}{\pi^{1/2}(-n+s-\frac{3}{2})!(-n+2s-1)!}}. \tag{32}$$

Now we address the problem of finding the creation and annihilation operator for the wave functions (30) with the factorization method [16], namely we intend to find differential operators $\hat{P}_{\pm}$ with property

$$\hat{P}_{\pm}\psi_n(u)=p_{\pm}\psi_{n\pm1}. \tag{33}$$

The operators (33) can be found with the act of the differential operator $\frac{d}{du}$ on the wave function (30),

$$\frac{d\psi_n(u)}{du}=N_n(-\epsilon u)(1-u^2)^{\epsilon/2-1}C_n^{-n+s}(u)+N_n(1-u^2)^{\epsilon/2}\frac{dC_n^{-n+s}(u)}{du}. \tag{34}$$

Here by using the recurrence relation for the Gegenbauer polynomial as

$$\frac{dC_n^{\lambda}(u)}{du}=2\lambda C_{n-1}^{\lambda+1}(u), \tag{35}$$

and by substituting the above relation in (34) we obtain

$$\frac{d\psi_n(u)}{du}=\frac{-\epsilon u}{1-u^2}\psi_n(u)+\frac{N_n}{N_{n-1}}\frac{2(s-n)}{(1-u^2)^{\frac{1}{2}}}\psi_{n-1}(u). \tag{36}$$

So, corresponding to (33), we have

$$\sqrt{1-u^2}\left[\frac{d}{du}+\frac{u(s-n-\frac{1}{2})}{1-u^2}\right]\psi_n(u)=\left[\frac{N_n}{N_{n-1}}2(s-n)\right]\psi_{n-1}(u). \tag{37}$$

From the above equation and (32), we can define the annihilation operator $\hat{P}_-$ as

$$\hat{P}_-=\sqrt{1-u^2}\left[\frac{d}{du}+\frac{u\epsilon}{1-u^2}\right]\sqrt{\frac{\epsilon+1}{\epsilon}}, \tag{38}$$

where

$$\epsilon = s - n - \frac{1}{2}. \tag{39}$$

The corresponding eigenvalue of $\hat{P}_-$ is as:

$$p_- = \sqrt{n(-n+2s)} = \sqrt{n\left(-n + \sqrt{1 + \frac{4\beta(\beta+1)}{q}}\right)}. \tag{40}$$

Again, by using the recurrence relation for the Gegenbauer polynomial as

$$2(\lambda-1)(2\lambda-1)uC_n^{\lambda}(u) = 4\lambda(\lambda-1)(1-u^2)C_{n-1}^{\lambda+1}(u) + (2\lambda+n-1)(n+1)C_{n+1}^{\lambda-1}(u), \tag{41}$$

and by substituting the above relation in (34), we can write

$$\begin{aligned}\frac{d\psi_n(u)}{du} &= \frac{-\epsilon u}{1-u^2}\psi_n + \frac{u(-2n+2s-1)}{(1-u^2)}\psi_n(u) \\ &\quad + N_n \frac{(-n+2s-1)(n+1)}{2(-n+s-1)}(1-u^2)^{\epsilon/2-1}C_{n+1}^{-n+s-1}(u).\end{aligned} \tag{42}$$

Then corresponding to (33), we have

$$\sqrt{1-u^2}\left[-\frac{d}{du} + \frac{u\epsilon}{1-u^2}\right]\psi_n(u) = \frac{N_n}{N_{n+1}} \frac{(-n+2s-1)(n+1)}{2(-n+s-1)}\psi_{n+1}(u). \tag{43}$$

From the above equation and (32), we can obtain the creation operator $\hat{P}_+$ as

$$\hat{P}_+ = \sqrt{1-u^2}\left[-\frac{d}{du} + \frac{u\epsilon}{1-u^2}\right]\sqrt{\frac{-n+s-\frac{3}{2}}{-n+s-\frac{1}{2}}}, \tag{44}$$

with following eigenvalue

$$p_+ = \sqrt{(n+1)(-n+2s-1)} = \sqrt{(n+1)\left(-n + \sqrt{1 + \frac{4\beta(\beta+1)}{q}} - 1\right)}. \tag{45}$$

Now, we obtain the algebra associated to the operators $\hat{P}_-$, $\hat{P}_+$. Using (38) and (34), we calculate the commutator $[\hat{P}_+, \hat{P}_-]$ as:

$$[\hat{P}_+, \hat{P}_-]\psi_n(u) = 2\hat{P}_0\psi_n(u), \tag{46}$$

where we define the operator $\hat{P}_0$ as

$$\hat{P}_0 = \hat{n} - s + \frac{1}{2} = \hat{n} - \sqrt{\frac{1}{4} + \frac{\beta(\beta+1)}{q}} + \frac{1}{2}. \tag{47}$$

The operators $\hat{P}_+$, $\hat{P}_-$ and $\hat{P}_0$ that are q-deformed operators, satisfy the following commutation relations:

$$[\hat{P}_0, \hat{P}_+] = \hat{P}_+, \qquad [\hat{P}_0, \hat{P}_-] = -\hat{P}_-, \qquad [\hat{P}_+, \hat{P}_-] = 2\hat{P}_0. \tag{48}$$

Obviously (48) show that these operators following commutation relations satisfy q-deformed su(2) Lie algebra. In other words our interesting problem has the SU(2) dynamical group.

4 Generalized Coherent States for q-Deformed Hyperbolic PT Potential

The Klauder-Perelomov definition of coherent states consists of applying the operator $e^{\xi \hat{P_+}}$ on the ground state $|0\rangle$, such that [4–6],

$$|\xi\rangle = e^{\xi \hat{P_+} - \bar{\xi}\hat{P_-}}|0\rangle. \tag{49}$$

The generalized coherent state based on su(2) is as:

$$|\zeta\rangle = e^{\zeta \hat{P_+}}\, e^{\log(1+|\zeta|^2)\hat{P_0}} e^{\bar{\zeta}\hat{P_-}}|0\rangle, \tag{50}$$

where $\zeta = \frac{\xi \tanh|\xi|}{|\xi|}$ is a complex number satisfying the condition $|\zeta| < 1$. The last equation can be reorganized as

$$|\zeta\rangle = (1+|\zeta|^2)^{\frac{1}{2}-s}\sum_{n=0}^{2s}\frac{\zeta^n (P_+)^n}{n!}|n\rangle. \tag{51}$$

By acting $\hat{P}_+$ (q-deformed operator) on $|n\rangle$, we obtain

$$|\zeta\rangle = (1+|\zeta|^2)^{\frac{1}{2}-s}\sum_{n=0}^{2s}\sqrt{\frac{{}_n(2s-1)}{n!}}\zeta^n|n\rangle, \tag{52}$$

in another term

$$|\zeta\rangle = (1+|\zeta|^2)^{\frac{1}{2}-\sqrt{\frac{1}{4}+\frac{\beta(\beta+1)}{q}}}\sum_{n=0}^{2s}\sqrt{\frac{{}_n(\sqrt{1+\frac{4\beta(\beta+1)}{q}}-1)}{n!}}\zeta^n|n\rangle, \tag{53}$$

where ${}_n(2s-1) = (2s-1)(2s-2)\dots(2s-n+1)$. Equation (53) is the generalized coherent states for q-deformed hyperbolic PT potential. In the coordinate representation, we will have

$$\langle x|\zeta\rangle = \psi_\zeta(x) = (1+|\zeta|^2)^{\frac{1}{2}-s}\sum_{n=0}^{2s}\sqrt{\frac{{}_n(2s-1)}{n!}}\zeta^n\psi_n(x). \tag{54}$$

In (53), if we consider for the case $q = 1$, we obtain the corresponding relations for operators, eigenvalues and SU(2) group of the ordinary hyperbolic PT potential [13, 14].

5 Conclusion

In this work, we have introduced generalized coherent states for SU(2) group using the Klauder-Perelomov definition of coherent states. Thereafter, in Sect. 3 we could calculate the Schrödinger equation for q-deformed hyperbolic PT potential that it is shown for the first time by Arai [17, 18], and here we have obtained the corresponding eigenvalues and eigenfunction. Then we established the ladder operators directly from the eigenfunction (30), using the factorization method. It is shown that these operators satisfy q-deformed su(2) Lie algebra. In Sect. 4, we have computed the generalized coherent states for q-deformed hyperbolic PT potential. We have shown that, for the case $q = 1$, we can obtain the same generalized coherent states for usual hyperbolic PT potential [13, 14].

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